Applied Mathematics Letters 24 (2011) 1455–1460

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Existence of oscillatory solutions of forced second order delay differential equations☆

Jurang Yan

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, People's Republic of China

ARTICLE INFO

Article history:
Received 22 September 2009
Received in revised form 15 March 2011
Accepted 16 March 2011

Keywords:
Second order delay differential equation
Oscillation
Global existence

ABSTRACT

In this paper, under weaker hypothesis, the global existence of oscillatory solutions is established for a forced second order nonlinear delay differential equation. The results of this paper generalize and improve some known results.

1. Introduction

In this paper, we are concerned with the existence of oscillatory solutions of forced second order nonlinear delay differential equations. During the past three decades, the investigation of oscillatory theory for delay differential equations and delay dynamic equations has attracted attention of numerous researchers due to their significance in theory and applications. We refer, in particular, to the monographs by Gyori and Ladas [1], Erbe et al. [2] and the article by Anderson and Zafer [3] and references therein. But, to the best of our knowledge the existence of oscillatory solutions for the delay or non-delay differential equation has been scarcely investigated [4–6].

The purpose of this paper is to study the existence of oscillatory solutions for the forced delay differential equation

$$y''(t) + a(t)y'(t) + \sum_{i=1}^{m} f_i(t, y(g_i(t))) = q(t), \quad t \geq t_0, \tag{1.1}$$

where a, q and $g_i \in C([t_0, \infty), R)$, $f_i \in C([t_0, \infty) \times R, R)$ and $g_i(t) \leq t$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \ldots, m$.

Let $t_{-1} = \min_{1\leq i\leq m} \inf_{t\geq t_0}\{g_i(t)\}$. By a solution of (1.1) we mean a function $y \in C([t_{-1}, \infty), R)$ for an initial condition $\phi \in C([t_{-1}, t_0], R)$, $y(t) = \phi(t)$, $t \in [t_{-1}, t_0]$ and satisfies (1.1) for $t \geq t_0$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise it is called non-oscillatory.

In the mathematical ecology, (1.1) includes various models. For example, the differential equation

$$y''(t) + a(t)y'(t) + b(t)\sin y(t - \tau) = 0,$$

was introduced in 1967 by Israelsson and Johnsson [7] as a model for the geotropic circumnutations of Helianthus annuus. The particular case of (1.1) is the ordinary differential equation

$$y'' + a(t)y' + f(t, y) = q(t), \quad t \geq t_0, \tag{1.2}$$

☆ This work was supported by the Natural Science Foundation of China (No. 11001157), Tianyuan Mathematics Fund of China (No. 10826080) and the Youth Science Foundation of Shanxi Province (No. 2009021001-1, No. 2010021001-1).

E-mail address: jryan@sxu.edu.cn.

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.03.031

where $a, q \in C([t_0, \infty), R), f \in C([t_0, \infty), R)$. When $a(t) \equiv 0$ and $f(t, y) = p(t)y$, (1.2) reduces respectively to

$$y'' + f(t, y) = q(t), \quad t \geq t_0 \tag{1.3}$$

and

$$y'' + p(t)y = q(t), \quad t \geq t_0, \tag{1.4}$$

where $p \in C([t_0, \infty), R)$. The existence of oscillatory solutions of (1.3) and (1.4) have been established in [4].

In this paper, using a modification of Rogovchenko's technique [4] and the well-known Schauder–Tychonoff fixed point theorem, we establish, under weaker hypothesis, a global existence theorem of oscillatory solutions of (1.1). Our results generalize and improve those in [4].

Throughout this paper, we will use the following notations.

For a constant $\gamma > 0$

$$p_i(t)_\gamma = \max_{|y| \leq \gamma} \frac{1}{\gamma} |f_i(t, y)|, \quad t \geq t_0, i = 1, 2, \ldots, m \tag{1.5}$$

$$p(t)_\gamma = \max_{|y| \leq \gamma} |f(t, y)|, \quad t \geq t_0$$

and define

$$h(t) = \exp\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right), \quad t \geq t_0.$$

2. Statement of the main results

In this section, we state main results in this paper.

Theorem 2.1. *Assume that there exists a constant $k > 0$ and sufficiently large a $T \geq t_0$ such that for any $T_1, T_2 \geq T$*

$$\left|\int_{T_1}^{T_2} a(t)\mathrm{d}t\right| \leq k, \tag{2.1}$$

$$\frac{1}{h(t)} \int_t^\infty h(s)q(s)\mathrm{d}s \text{ is integrable on } [t_0, \infty), \tag{2.2}$$

and there exists $\gamma > 0$ such that

$$\frac{1}{h(t)} \int_t^\infty h(s) \sum_{i=1}^{m} p_i(s)_\gamma \mathrm{d}s \text{ is integrable on } [t_0, \infty). \tag{2.3}$$

Moreover, there exist two increasing divergent sequences $\{t_n\}$ and $\{s_n\}$, $t_n, s_n \geq t_0$ such that

$$\int_{t_n}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) + \gamma \sum_{i=1}^{m} p_i(\tau)_\gamma\right] \mathrm{d}\tau \mathrm{d}s > 0 \tag{2.4}$$

and

$$\int_{s_n}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) - \gamma \sum_{i=1}^{m} p_i(\tau)_\gamma\right] \mathrm{d}\tau \mathrm{d}s < 0, \quad \text{for all } n \geq 1. \tag{2.5}$$

Then there exists $T_\gamma > t_0$ so that (1.1) has an oscillatory solution $y(t)$ defined on $[T'_\gamma, \infty)$ with $|y(t)| \leq \gamma$, $t \geq T_\gamma$ and $\lim_{t\to\infty} y(t) = 0$ where $T'_\gamma = \min_{1\leq i\leq m} \inf_{t\geq T_\gamma} \{g_i(t)\}$.

Corollary 2.1. *Assume that (2.1)–(2.3) hold. If*

$$\liminf_{t\to\infty} \frac{\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^{m} p_i(\tau)_\gamma \mathrm{d}\tau\mathrm{d}s} < 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^{m} p_i(\tau)_\gamma \mathrm{d}\tau\mathrm{d}s} > 0, \tag{2.6}$$

then the conclusion of Theorem 2.1 is valid.

Consider differential equations (1.3) and (1.4). By Corollary 2.1 we immediately obtain the following two results.

Corollary 2.2. *Assume that*

$$\int_{t_0}^{\infty} tq(t)\mathrm{d}t \text{ exists and is finite}, \tag{2.7}$$

and there is a $\gamma > 0$ such that $\int_{t_0}^{\infty} tp(t)_\gamma \mathrm{d}t < \infty$. If

$$\liminf_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty p(\tau)_\gamma\mathrm{d}\tau\mathrm{d}s} < 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty p(\tau)_\gamma\mathrm{d}\tau\mathrm{d}s} > 0,$$

then there is a $T_\gamma > t_0$ such that (1.3) has an oscillatory solution $y(t)$ defined on $[T_\gamma, \infty)$ with $|y(t)| \le \gamma$, $t \ge T_\gamma$ and $\lim_{t\to\infty} y(t) = 0$.

Corollary 2.3. *Assume that (2.7) holds and $\int_{t_0}^{\infty} t|p(t)|\mathrm{d}t < \infty$. If*

$$\liminf_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty |p(\tau)|\mathrm{d}\tau\mathrm{d}s} < 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty |p(\tau)|\mathrm{d}\tau\mathrm{d}s} > 0,$$

then (1.4) has an oscillatory solution $y(t)$ defined on $[t_0, \infty)$ and $\lim_{t\to\infty} y(t) = 0$.

Remark 2.1. Corollary 2.2 improves the main result in [4, Theorem 3.1] which requires conditions $\int_{t_0}^{\infty} t|q(t)|\mathrm{d}t < \infty$ and

$$\liminf_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty p(\tau)_\gamma\mathrm{d}\tau\mathrm{d}s} < -1 \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty p(\tau)_\gamma\mathrm{d}\tau\mathrm{d}s} > 1,$$

in order that (1.3) has an oscillatory solution $y(t)$ defined on $[T_\gamma, \infty)$ with $\lim_{t\to\infty} y(t) = 0$.

Remark 2.2. Corollary 2.3 improves Corollary 3.3 in [4] which requires conditions $\int_{t_0}^{\infty} t^2|p(t)|\mathrm{d}t < \infty$, $\int_{t_0}^{\infty} t|q(t)|\mathrm{d}t < \infty$,

$$\liminf_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty |p(\tau)|\mathrm{d}\tau\mathrm{d}s} = -\infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(\tau)\mathrm{d}\tau\mathrm{d}s}{\int_t^\infty \int_s^\infty |p(\tau)|\mathrm{d}\tau\mathrm{d}s} = \infty,$$

in order that (1.4) has an oscillatory solution $y(t)$ defined on $[t_0, \infty)$ with $\lim_{t\to\infty} y(t) = 0$.

Remark 2.3. The results of this paper, as in [4], does not require monotonicity assumptions on nonlinear function $f_i(t, u)$ and $f_i(t, u)$ does not have to satisfy the sign condition for all $t \ge t_0$, that is, $uf_i(t, u) > 0$ for $u \ne 0$.

3. Proof of the main results

Proof of Theorem 2.1. Our proof is based on an application of the well-known Schauder–Tychonoff fixed point theorem. From (2.2) and (2.3), for any $\gamma > 0$ we can choose a large $T_\gamma \ge T$ such that for all $t \ge T_\gamma$

$$\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \left[q(\tau) + \gamma \sum_{i=1}^m p_i(\tau)_\gamma \right] \mathrm{d}\tau\mathrm{d}s \le \gamma \tag{3.1}$$

and

$$\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \left[q(\tau) - \gamma \sum_{i=1}^m p_i(\tau)_\gamma \right] \mathrm{d}\tau\mathrm{d}s \ge -\gamma. \tag{3.2}$$

Let $C[T'_\gamma, \infty)$ denote the locally convex space of all continuous functions y on $[T'_\gamma, \infty)$ endowed with the topology of uniform convergence on compact subsets of $[T'_\gamma, \infty)$.

Define a set S as follows:

$$S = \{y \in C[T'_\gamma, \infty) : |y| \le \gamma\}.$$

Observe that S is a nonempty closed convex subset of $C[T'_\gamma, \infty)$. Define an operator F by

$$(Fy)(t) = \begin{cases} \int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \left[q(\tau) - \sum_{i=1}^m f_i(\tau, y(g_i(\tau))) \right] \mathrm{d}\tau\mathrm{d}s, & t \ge T_\gamma, \\ (Fy)(T_\gamma), & T'_\gamma \le t \le T_\gamma. \end{cases} \tag{3.3}$$

It is easy to see from (2.2) and (2.3) that for any $y \in S$, $(Fy)(t)$ is well defined on $[T'_\gamma, \infty)$. Thus, in view of (3.1) and (3.2), for all $t \geq T'_\gamma$,

$$(Fy)(t) \leq \int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) + \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right] \mathrm{d}\tau\,\mathrm{d}s \leq \gamma \tag{3.4}$$

and

$$(Fy)(t) \geq \int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) - \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right] \mathrm{d}\tau\,\mathrm{d}s \geq -\gamma. \tag{3.5}$$

Hence, from (3.4) and (3.5), for any $y \in S$

$$|(Fy)(t)| \leq \gamma, \quad t \geq T'_\gamma, \tag{3.6}$$

which implies that $FS \subset S$ and Fy is uniformly bounded on S.

The continuity of $F : S \to S$ is verified as follows: let $\{y_n\}_{n=1}^\infty \subset S$ be any sequence and $y_0 \in S$ with $\lim_{n\to\infty} y_n = y_0$. For any $\epsilon > 0$, let $T_1 > T_\gamma$ be a fixed number so that

$$\int_{T_1}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^m p_i(\tau)_\gamma \mathrm{d}\tau\,\mathrm{d}s < \frac{\epsilon}{3\gamma}. \tag{3.7}$$

From the compactness of the domain of f_i, there exists a large $N > 0$ and a constant $\delta(\epsilon) > 0$. Let $t \in [T'_\gamma, T_1]$ and $n \geq N$ when $|y_n - y_0| < \delta(\epsilon)$,

$$\max_{1\leq i\leq m} |f_i(t, y_n(g_i(t))) - f_i(t, y_0(g_i(t)))| \leq \frac{\epsilon}{3m(T_1 - T'_\gamma)^2}, \tag{3.8}$$

where y satisfies $|y - y_0| < \delta(\epsilon)$. By virtue of (3.3)–(3.8), we have that for any $t \geq T'_\gamma$ and $|y_n - y_0| < \delta$

$$\begin{aligned}
|(Fy_n)(t) - (Fy_0)(t)| &\leq \int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^m |f_i(\tau, y_n(g_i(\tau))) - f_i(\tau, y_0(g_i(\tau)))| \mathrm{d}\tau\,\mathrm{d}s \\
&\leq \int_{T'_\gamma}^{T_1} \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^m |f_i(\tau, y_n(g_i(\tau))) - f_i(\tau, y_0(g_i(\tau)))| \mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{T_1}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^m |f_i(\tau, y_n(g_i(\tau))) - f_i(\tau, y_0(g_i(\tau)))| \mathrm{d}\tau\,\mathrm{d}s \\
&\leq \int_{T'_\gamma}^{T_1} (s - T'_\gamma) \sum_{i=1}^m |f_i(s, y_n(g_i(s))) - f_i(s, y_0(g_i(s)))| \mathrm{d}s \\
&\quad + 2\gamma \int_{T_1}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau) \sum_{i=1}^m p_i(\tau)_\gamma \mathrm{d}\tau\,\mathrm{d}s < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon,
\end{aligned}$$

where (2.1) and (1.5) are used. The continuity of F on S is proved.

Moreover, in view of (2.1)–(2.3) and (3.3) there exists a positive constant M such that for all $t \geq T_\gamma$

$$|(Fy)'(t)| \leq \left|\frac{1}{h(t)} \int_t^\infty h(s)q(s)\mathrm{d}s\right| + \left|\frac{\gamma}{h(t)} \int_t^\infty h(s) \sum_{i=1}^m p_i(s)_\gamma \mathrm{d}s\right| \leq M. \tag{3.9}$$

From (3.9) it is easy to see that $(Fy)'(t)$, $y \in S$, is uniformly bounded for t on $[T'_\gamma, \infty)$. This implies Fy is equicontinuous on S. Therefore F maps S continuously into a compact subset of S. Consequently, by the Schauder–Tychonoff fixed point theorem, F has a fixed point y^* in S, that is,

$$(Fy^*)(t) = y^*(t) = \begin{cases} \displaystyle\int_t^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) - \sum_{i=1}^m f_i(\tau, y^*(g_i(\tau)))\right] \mathrm{d}\tau\,\mathrm{d}s, & t \geq T_\gamma, \\ (Fy^*)(T_\gamma), & T'_\gamma \leq t \leq T_\gamma. \end{cases} \tag{3.10}$$

On the other hand, from (2.4) and (2.5), we find

$$y^*(t_n) \leq \int_{t_n}^\infty \frac{1}{h(s)} \int_s^\infty h(\tau)\left[q(\tau) + \gamma \sum_{i=1}^m p_i(\tau)_\gamma\right] \mathrm{d}\tau\,\mathrm{d}s \leq 0$$

and

$$y^*(s_n) \geq \int_{s_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) \left[q(\tau) - \gamma \sum_{i=1}^{m} p_i(\tau)_\gamma \right] \mathrm{d}\tau \mathrm{d}s \geq 0, \quad \text{for all } n \geq 1.$$

Hence $y^*(t)$ is an oscillatory solution of (1.1) and $\lim_{t\to\infty} y^*(t) = 0$. In view of (2.4) and (2.5) $y^*(t)$ is not identically equal to 0 on any infinite subinterval of $[T_\gamma, \infty)$. The proof of Theorem 2.1 is complete. □

The Proof of Corollary 2.1. From (2.6), there exists a positive constant $\gamma_0 \leq \gamma$ and two increasing divergent sequences $\{t_n\}$ and $\{s_n\}$ so that

$$\begin{aligned}\int_{t_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) q(\tau) \mathrm{d}\tau \mathrm{d}s &\leq -\gamma_0 \int_{t_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) \sum_{i=1}^{m} p_i(\tau)_\gamma \mathrm{d}\tau \mathrm{d}s \\ &\leq -\gamma_0 \int_{t_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) \sum_{i=1}^{m} p_i(\tau)_{\gamma_0} \mathrm{d}\tau \mathrm{d}s\end{aligned}$$

and

$$\int_{s_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) q(\tau) \mathrm{d}\tau \mathrm{d}s \geq \gamma_0 \int_{s_n}^{\infty} \frac{1}{h(s)} \int_s^{\infty} h(\tau) \sum_{i=1}^{m} p_i(\tau)_{\gamma_0} \mathrm{d}\tau \mathrm{d}s, \quad \text{for all } n \geq 1,$$

that is, (2.4) and (2.5) with $\gamma = \gamma_0$ are satisfied. Hence all the conditions of Theorem 2.1 hold. The proof of corollary is complete. □

4. Examples

In this section, we consider some illustrative examples which imply that the known results existing in the literature cannot be applied to these examples.

Example 1. Consider a linear second order equation with a forced term

$$y'' + p(t)y = q(t), \quad t \geq t_0, \tag{4.1}$$

where $p(t) = ke^{-t}$, $q(t) = -2e^{-t}\cos t + ke^{-2t}\sin t$, $0 < k < 1$. Thus

$$\begin{aligned}Q(t) &:= \int_t^{\infty} \int_s^{\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s = e^{-t}\sin t + \frac{4k}{25}e^{-2t}\cos t - \frac{3k}{25}e^{-2t}\sin t, \\ P(t) &:= \int_t^{\infty} \int_s^{\infty} p(\tau)_{\gamma=1}\mathrm{d}\tau\mathrm{d}s = \int_t^{\infty} \int_s^{\infty} \max_{|y|\leq 1} ke^{-\tau}|y|\mathrm{d}\tau\mathrm{d}s = ke^{-t}.\end{aligned}$$

Let $t_n = (2n - \frac{1}{2})\pi$, $s_n = (2n + \frac{1}{2})\pi$, $n = 1, 2, \ldots$.

$$Q(t_n) + P(t_n) = e^{-t_n}\sin t_n \left(1 - \frac{3k}{25}e^{-t_n}\right) + \frac{4k}{25}e^{-2t_n}\cos t_n + ke^{-t_n} = -e^{-t_n}\left(1 - \frac{3k}{25}e^{-t_n} - k\right), \tag{4.2}$$

$$Q(s_n) - P(s_n) = e^{-s_n}\sin s_n \left(1 - \frac{3k}{25}e^{-s_n}\right) + \frac{4k}{25}e^{-2s_n}\cos s_n - ke^{-s_n} = e^{-s_n}\left(1 - \frac{3k}{25}e^{-s_n} - k\right). \tag{4.3}$$

It is easy to see from (4.2) and (4.3) that there exists a large $N \geq 1$ such that for all $n \geq N$

$$Q(t_n) + P(t_n) \leq 0 \quad \text{and} \quad Q(s_n) - P(s_n) \geq 0.$$

Thus by Theorem 2.1, (4.1) has an oscillatory solution $y(t)$ on $[t_0, \infty)$ and $\lim_{t\to\infty} y(t) = 0$. It is not difficult to check that (4.1) has the oscillatory solution $y(t) = e^{-t}\sin t$.

On the other hand, by [8, Theorem 1A] we know that for any constants $c_0, c_1 \in R$, (4.1) has a solution $y(t)$ on $[T, \infty)$, where $T \geq \max\{t_0, 1\}$ depends on c_0 and c_1, which is asymptotic to the line $c_0 + c_1 t$ for $t \to \infty$, i.e.

$$y(t) = c_0 + c_1 t + \circ(1) \quad \text{for } t \to \infty \tag{4.4}$$

and, in addition, satisfies

$$y'(t) = c_1 + \circ(1) \quad \text{for } t \to \infty. \tag{4.5}$$

Consequently we have proved that (4.1) has both bounded oscillatory solutions tending to zero as $t \to \infty$ and unbounded solutions tending to infinity satisfying (4.4) and (4.5).

Example 2. Consider the forced Emden–Fowler equation with delay

$$y''(t) + p(t)y^{\sigma}(t - 2\pi) = q(t), \quad t \geq t_0, \tag{4.6}$$

where σ is the quotient of two odd integers, $p(t) = e^{-t}$, $q(t) = 2e^{-t}\sin t + e^{-(1+\sigma)t-2\pi\sigma}\cos^{\sigma} t$. Taking into account

$$q(t) = 2e^{-t}\sin t + \circ(e^{-t}) \quad \text{and} \quad p(t)_{\gamma} = e^{-t}\gamma^{\sigma-1}$$

we obtain

$$Q(t) := \int_t^{\infty}\int_s^{\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s = e^{-t}(\sin t - \cos t) + \circ(e^{-t}),$$

$$P(t) := \int_t^{\infty}\int_s^{\infty} p(\tau)_{\gamma}\mathrm{d}\tau\mathrm{d}s = \int_t^{\infty}\int_s^{\infty} e^{-\tau}\gamma^{\sigma}\mathrm{d}\tau\mathrm{d}s = \gamma^{\sigma-1}e^{-t}.$$

Thus

$$\liminf_{t\to\infty}\frac{Q(t)}{P(t)} = -\frac{1}{\gamma^{\sigma-1}} < 0 \quad \text{and} \quad \limsup_{t\to\infty}\frac{Q(t)}{P(t)} = \frac{1}{\gamma^{\sigma-1}} > 0.$$

By Corollary 2.1, (4.6) has a bounded oscillatory solution $y(t)$ and $\lim_{t\to\infty} y(t) = 0$. Indeed, $y(t) = e^{-t}\cos t$ is such a solution.

Example 3. Consider the following differential equation

$$y'' + p(t)f(y) = q(t), \quad t \geq t_0 > 0, \tag{4.7}$$

where $f \in C(R, R)$, $p(t) = \frac{2}{t^3}$, $q(t) = -\frac{\sin t}{t} - 2\frac{\cos t}{t^2} + 2\frac{\sin t}{t^3}$.

$$Q(t) := \int_t^{\infty}\int_s^{\infty} q(\tau)\mathrm{d}\tau\mathrm{d}s = \frac{\sin t}{t}, \qquad P(t) := \int_t^{\infty}\int_s^{\infty} p(\tau)_{\gamma}\mathrm{d}\tau\mathrm{d}s = \frac{M_{\gamma}}{t},$$

where $M_{\gamma} = \max_{|y|\leq\gamma}\frac{1}{\gamma}|f(y)|$. Thus

$$\liminf_{t\to\infty}\frac{Q(t)}{P(t)} = -\frac{1}{M_{\gamma}} < 0 \quad \text{and} \quad \limsup_{t\to\infty}\frac{Q(t)}{P(t)} = \frac{1}{M_{\gamma}} > 0.$$

By Corollary 2.2, we conclude that (4.7) has an oscillatory solution $y(t)$ defined on $[t_0, \infty)$ with $\lim_{t\to\infty} y(t) = 0$.

Example 4. Consider the following nonlinear second order equation with damping term,

$$y'' + a(t)y' + p(t)|y^{\lambda-1}|y = q(t), \tag{4.8}$$

where $\lambda > 1$ is a constant, $a(t) = e^{-\alpha t}$, $p(t) = e^{-\beta t}\sin t$, $q(t) = e^{-\sigma t}\cos t$, $0 < \alpha < 1$, $\beta > 1$, $\sigma > 1$ are constants.

It is straightforward to verify that the conditions (2.1)–(2.3) and (2.6) are satisfied. Hence by Corollary 2.1, Eq. (4.8) has at least one oscillatory solution $y(t)$ with $\lim_{t\to\infty} y(t) = 0$.

Acknowledgements

The author thanks both referees for a careful reading of the manuscript and useful suggestions that helped to improve the presentation.

References

[1] I. Gyori, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon, Oxford, 1991.
[2] L.H. Erbe, Q. Kong, B.G. Zhang, Oscillation Theorem for Functional Differential Equations, Marcel Dekker, New York, 1995.
[3] D.R. Anderson, A. Zafer, Nonlinear oscillation of second-order dynamic equations on time scales, Appl. Math. Lett. 22 (2009) 1591–1597.
[4] O.M. Mustafa, Yu.V. Rogovchenko, Oscillation of second-order perturbed differential equation, Math. Nachr. 278 (2005) 460–469.
[5] C.H. Ou, J.S.H. Wong, On existence of oscillatory solutions of second order Emden–Fowler equations, J. Math. Anal. Appl. 277 (2003) 670–680.
[6] D.W. Willett, On the oscillatory behavior of the solutions of second order linear differential equations, Ann. Polon. Math. 21 (1969) 175–191.
[7] D. Israelsson, A. Johnsson, A theory of circumnutatins of Helianthus annus, Physiol. Plants 20 (1967) 957–976.
[8] Ch.G. Philos, I.K. Purnaras, P.Ch. Tsamatos, Asymptotic to polynomimals solutions for nonlinear differential equations, Nonlinear Anal. 59 (2004) 1157–1179.